QUASICONFORMAL MAPPINGS AND PROBLEMS IN HYDRODYNAMICS

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In the present study the authors faced a difficult task, that of clarifying M. A. Lavrent'ev's profound ideas on the mathematical problems of hydrodynamics to his students and followers over the past decade. The period up to 1970 is reflected in full detail in a survey prepared by the editorial board of the anniversary collection [1] published on the occasion of Mikhail Alekseevich's 70th birthday and in his excellent book [2] written in collaboration with B. V. Shabat, which has already gone through two editions. In recent years, some of the problems posed by M. A. Lavrent'ev and touched upon in the aforementioned publications have been fully or partially solved. A description of these (partially unpublished) results, obtained chiefly by members of the staff at the Hydrodynamics Institute of the Siberian Branch of the Academy of Sciences of the USSR, a creation of Mikhail Alekseevich, constitutes the substance of this study.

1. L-Elliptic Systems. In 1946 [3, 4] Lavrent'ev drew attention for the first time to a class of nonlinear systems of equations which today are usually called strongly elliptic in Lavrent'ev's sense (L-elliptic)

$$F(z, w, w_z, w_{\bar{z}}) \equiv f_1 + if_2 = 0 \quad (z = x + iy,\ w = \varphi + i\psi), \tag{1.1}$$

characterized by the property that every bounded solution $w = w(z)$ ($z \in D$) is locally homeomorphic in D. The importance of the study of L-elliptic systems of equations is due, in particular, to their direct hydrodynamic interpretation: they describe many complex hydrodynamic processes, such as subsonic potential stationary flows of an ideal gas, filtration of a liquid in nonuniform anisotropic porous media or of a liquid with a nonlinear law of motion, and others.

On the basis of hydrodynamic considerations, Lavrent'ev describes (1.1) in the form of equations in the geometric characteristics p_s, α_s, h_s, and θ_s a parallelogram with a vertex at the point z_0 (p_s, length of a side; α_s, angle it forms with the axis OX; h_s, altitude opposite the angle θ_s at z_0), which is transformed by the tangent mapping

$$w = w(z_0) + w_z(z_0)(z - z_0) + w_{\bar{z}}(\bar{z} - \bar{z}_0), \quad J = |w_z|^2 - |w_{\bar{z}}|^2$$

into a unit square inclined at an angle s to the axis $\varphi = \operatorname{Re} w$. The derivatives w_z and $w_{\bar{z}}$ can be expressed in an elementary manner in terms of $H_s = h_s + i\theta_s$ and $P_s = p_s + i\alpha_s$ and, after substitution into (1.1), lead to an equation in the characteristics which is assumed to be solvable for H_s:

$$H_s = H_s(z, w, P_s) \in C^1(\Omega),\ \forall s \in [0, 2\pi].$$

According to [3], Eq. (1.1) is L-elliptic if there exists a constant $\delta > 0$ such that

$$\delta < \theta_s < 2\pi - \delta,\ \delta < \partial h_s/\partial p_s < \delta^{-1},\ \forall s \in [0, 2\pi],$$

i.e., the characteristic parallelogram is not degenerate.

Lavrent'ev showed [4, 5] that the solutions of L-elliptic equations possess many properties of conformal mappings and, in particular, satisfy an analog of Riemann's theorem on the mapping of simply connected domains. Subsequently, the theory of L-elliptic equations was further developed in the works of Lavrent'ev and Shabat [6, 7], which, in addition to other interesting results, established some important properties of L-elliptic equations: the fact that they are elliptic in the ordinary sense and the fact that the quantity $|w_{\bar{z}}/w_z| \leqslant q_0 < 1$ is bounded.

In 1973 Monakhov [8] (see also [9, 10]) proposed using these properties as the basis for a definition and calling Eq. (1.1) L-elliptic if it is elliptic and solvable for ψ_x, ψ_y, as a result of which it can be written in the form

$$w_{\bar{z}} - q(z, w, \zeta) w_z = 0, \quad q \in C^1(\Omega) \quad (\zeta = \varphi_x - i\psi_y),$$

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subject to the requirement that $|q(z, w, \zeta)| \leqslant q_0 < 1$ in Ω. Monakhov established the equivalence of this definition of L-ellipticity to the geometric definition given by Lavrent'ev and proved an existence theorem for the mappings of multiply connected domains onto canonical regions by the solutions of L-elliptic equations. In 1974-1976, analogous results were obtained in the works of Bojarski and Iwaniec [11-13], and Kucher [14] studied boundary-value problems for L-elliptic equations.

In the proof given in [8] for the equivalence of the definitions of L-ellipticity it was shown that a direct consequence of uniform ellipticity of Eq. (1.1) will be the inequalities

$$|F_\zeta|^2 - |F_{\bar\zeta}|^2 \geqslant \alpha, \quad |F_\omega|^2 - |F_{\bar\omega}|^2 \geqslant \alpha > 0, \tag{1.2}$$

where $\zeta = \varphi_x - i\varphi_y$; $\omega = \psi_y + i\psi_x$. It was unexpectedly found that if the inequalities (1.2) hold, this guarantees not only local but global solvability of (1.1) for ω and ζ, which in turn ensures global solvability of the equation in the characteristics for H_s. Specifically, the following statement (Monakhov) holds:

THEOREM 1. Suppose that $F(0, 0) = 0$ and the function $F(\zeta, \omega)$ has continuous derivatives with respect to ζ and ω which satisfy the inequalities (1.2). Then there exists a continuously ddifferentiable homeomorphism $\omega(\zeta)$, $\omega: \mathbf{C} \to \mathbf{C}$, $\omega(t) = t = 0, \infty$ such that $F[\zeta, \omega(\zeta)] = 0$.

We shall sketch very briefly the proof of this result. From the uniform boundedness and from the inequalities (1.2), we have

$$|\mu_i(\zeta, \omega)| \leqslant \mu_0 < 1, \quad \mu_1 = F_{\bar\zeta}/F_\zeta, \quad \mu_2 = F_{\bar\omega}/F_\omega. \tag{1.3}$$

Formally differentiating the identity $F^\circ(\zeta) = F[\zeta, \omega(\zeta)] \equiv 0$ and substituting the values of the derivatives into the relation

$$F^0_{\bar\zeta} - \mu_1^0 F^0_\zeta = 0, \quad |\mu_1^0| = |\mu_1[\zeta, \omega(\zeta)]| \leqslant \mu_0 < 1, \tag{1.4}$$

Taking account of (1.3) for μ_1 and μ_2, we arrive at the equation

$$\omega_{\bar\zeta} - q_1(\zeta, \omega)\,\omega_\zeta + q_2(\zeta, \omega)\,\bar\omega_{\bar\zeta} = 0, \quad |q_1| + |q_2| \leqslant q_0 < 1, \tag{1.5}$$

where $q_i = \mu_i(1 - |\mu_j|^2)(1 - |\mu_1\mu_2|^2)^{-1}$, $i, j = 1, 2$, $i \neq j$. We arbitrarily fix $(\zeta_0, \omega_0) \neq (0, 0): F(\zeta_0, \omega_0) = 0$ from a neighborhood of the point (0, 0), where the equation $F(\zeta, \omega) = 0$ is locally solvable, and we shall consider $\zeta_0 = \omega_0 = 1$ (elongating ζ and ω if necessary). We shall use the term Q-automorphism to denote homeomorphic mappings $\omega = \omega(\zeta): \mathbf{C} \to \mathbf{C}$, $\omega(t) = t = 0.1, \infty$, for which $|\omega_{\bar\zeta}/\omega_\zeta| \leqslant$ const < 1, where $\{\omega, \omega^*\} \in W^1_p(E)$, $E\{|\zeta| < 1\}$, $p > 2$, $\omega^*(\zeta) = [\omega(1/\zeta)]^{-1}$.

We can convince ourselves that the Q-automorphism $\omega = \omega(\zeta)$ of Eq. (1.5) is the desired implicit function of the equation $F(\zeta, \omega) = 0$. Consequently the validity of Theorem 1 can be seen from the following assertion.

THEOREM 2. Let $q_i(\zeta, \omega)$ $(i = 1, 2)$ be continuous functions of (ζ, ω) in $\mathbf{C}^2$. Then there exists a Q-automorphism of Eq. (1.5).

This theorem is of interest in its own right, since previously, in obtaining similar results [15, 10], it was always assumed that the coefficients of Eq. (1.5) have a finite support with respect to $\zeta: q_i \equiv 0$, $|\zeta| > R$ (we can assume $R = 1$). Q-automorphisms of such equations (which we shall call Q_0-automorphisms) can be sought in the form

$$\omega = (1 + a)\zeta + a\bar\zeta - \frac{\zeta}{\pi}\iint_E \frac{f(t)\,dt}{t(t - \zeta)}, \quad \omega(1) = 1, \tag{1.6}$$

obtaining a nonlinear singular integral equation solvable in $L_p(E)$ $(p > 2)$ for the desired function $f(t)$. It can be verified that a Q-automorphism of Eq. (1.5) when $q_2 \equiv 0$ and q_1 is arbitrary can be represented in the form $\omega = 1/\omega^*(t)$ $(t = 1/\varphi(\zeta))$ in terms of Q_0-automorphisms of the equations

$$\varphi_{\bar\zeta} - \delta_1(\varphi)\, m_1(\zeta, \omega^*)\,\varphi_\zeta = 0, \quad \omega^*_{\bar t} - \delta_2(t)\, m_2(\zeta, \omega^*)\,\omega^*_t = 0 \quad (t = 1/\varphi), \tag{1.7}$$

where $m_1 = q_1(\zeta, 1/\omega^*)$; $m_2 = m_1\overline{\zeta_{\bar t}}/\zeta_t$; $\delta_1(\varphi) = 1$ for $|\varphi| \leqslant 1$; $\delta_1(\varphi) = 0$ for $|\varphi| > 1$, and $\delta_2(t) = 1 - \delta_1(1/t)$. If $q_2 \not\equiv 0$, then the Q-automorphism of Eq. (1.5) can be represented in implicit form in terms of the Q-automorphisms of $\sigma = \omega_1(\zeta)$ and $\sigma = \omega_2(\omega)$ of the equations

$$\partial\omega_i/\partial\bar\zeta_i - q_{1i}\partial\omega_i/\partial\zeta_i = 0, \quad |q_{1i}| \leqslant q_0 < 1,$$

$$q_{1i} = 2q_i\left(n_{ij} + \sqrt{n_{ij}^2 - 4q_i^2}\right)^{-1}, \quad n_{ij} = 1 + |q_i|^2 - |q_j|^2,$$

$$i, j = 1, 2, \; i \neq j, \; \zeta_1 = \zeta, \; \zeta_2 = \omega.$$

In an analogous manner, we set

$$\omega_i = 1/\omega_i^*(t_i) \quad (t_i = 1/\varphi_i(\zeta_i)),$$

where $\varphi_i(\zeta_i)$ and $\omega_i^*(t_i)$ are the Q_0-automorphisms of equations of the form (1.7) with coefficients defined by the previous formulas, in which all of the quantities have the subscript i. Thus, we see in this case also that the Q-automorphisms of Eq. (1.5) can be expressed in terms of Q_0-automorphisms, which with the use of the representations (1.6) for φ_i and ω_i^* leads to a system of four nonlinear singular equations solvable in $L_p(E)$ $(p > 2)$.

2. Two-Dimensional Problems of Subsonic Gasdynamics. The fundamental model of L-elliptic systems used by Lavrent'ev was the following system of equations:

$$x^k\rho(q)\varphi_x = \psi_y,\ x^k\rho(q)\varphi_y = -\psi_x,\ q^2 = |\nabla\varphi|^2,\ d(\rho q)/dq \geqslant \delta > 0, \tag{2.1}$$

describing plane (k = 0) and axisymmetric (k = 1) potential uniformly subsonic gas flows. Here φ and ψ are the potential and the stream function of the flow, respectively; ρ and q are the density and the magnitude of the velocity. From the results he obtained [3, 4] for L-elliptic equations it follows, in particular, that the plane problem of subsonic potential flows of a gas in a channel with curvilinear walls is solvable. The solvability of a somewhat more complicated problem of flow past a solid with a curved contour within the framework of the same model was established in 1954 by Bers (see [16]). A much more difficult problem was that of extending the known results found by Lavrent'ev [17] on the theory of jets of an incompressible liquid to the case of subsonic potential gas flows. A characteristic feature of jet problems in hydrodynamics, which makes their investigation much more difficult, is the fact that not only the solutions of the appropriate systems of equations but the domains of definition of the solutions as well are unknown. A broad class of hydrodynamic problems with free boundaries (jet and wave problems, problems in the projection of a wing profile on the basis of a known chord diagram, and others) can be described by the following boundary conditions, which must be satisfied by the solutions of the equations (2.1):

$$\partial\varphi/\partial n = 0 \text{ on } \Gamma;\ \theta = \theta(x) \text{ on } \Gamma_1;\ q = q(x, y) \text{ on } \Gamma_2, \tag{2.2}$$

where $\theta = \arctan(\varphi_y/\varphi_x)$ is the angle of inclination of the tangent to a given part Γ_1 of the boundary Γ of the region of flow, and $\Gamma_2 \subset \Gamma$ is a free (unknown) boundary. In 1963-1964 Monakhov, using a special choice of unknown variables (x and ψ), reduced the problems (2.1), (2.2) to boundary-value problems for a quasilinear degenerate elliptic system of equations in a given domain and, for certain conditions on $\theta(x)$ and $q(x, y)$, proved in [9] the solvability of these problems in the plane case. A detailed discussion of these results is given in [10], which also gives the generalizations, obtained in 1969, of theorems concerning the existence of solutions in some of the problems (2.1), (2.2) to the case of doubly connected regions (S. N. Antontsev) and to the case of eddying flows of a compressible liquid (P. I. Plotnikov). Axisymmetric problems of the type of (2.1), (2.2) were first studied in [18], and in Antontsev's book [19] these results were extended to the case of transonic gas flows, when degeneracy of the system (2.1) $(d(q\rho)/dq = 0)$ is allowed. In [19] Antontsev proved the finiteness of the propagation of perturbations in a number of jet problems (a consequence of the degeneracy of (2.1)), a fact first established by Ovsyannikov [20] for the problem of gas flow out of a vessel with rectilinear walls.

3. Problem of Conformal and Quasiconformal Gluing. Lavrent'ev [21] first formulated and solved the problem of conformal gluing, and he subsequently proposed the formulation of a general problem of quasiconformal gluing [22], which we shall state in the somewhat more general form commonly used today. Let D^+ be a finite simply or multiply connected domain with boundary $\Gamma = \partial D^+$ and $D^- = \mathbf{C}\setminus\overline{D^+}$. It is required to find homeomorphic solutions $\omega = \omega^\pm(z)$, $\omega^\pm$: $D^\pm \to \Delta^\pm$, $\overline{\Delta^+} \cup \Delta^- = \mathbf{C}$ of equations which are L-elliptic in $D^\pm$ and are related by the following gluing conditions for the images of the points t and $\alpha(t)$ lying on Γ:

$$\omega^+[\alpha(t)] = \omega^-(t),\ t \in \Gamma, \tag{3.1}$$

where $\alpha(t)$ is the homeomorphism $\Gamma \to \Gamma$. Conformal-gluing problems (i.e., problems in which $\omega^\pm$ are holomorphic in $D^\pm$) have been widely applied to the solution of more complicated boundary-value problem in the theory of analytic functions, for example, that of Carleman's problem, in which the desired functions $\Phi^\pm(z)$, $|\Phi^-(\infty)| < \infty$, which are holomorphic in $D^\pm$, satisfy on Γ the condition

$$\Phi^+[\alpha(t)] = a_0(t)\Phi^-(t) + a_1(t),\ a_0 \neq 0,\ t \in \Gamma. \tag{3.2}$$

We denote by $z^{\pm}:\Delta^{\pm} \to D^{\pm}$ the mappings which are inverse to the solution of the conformal-gluing problem (3.1) and set $\Phi_1^{\pm}(\omega) \equiv \Phi^{\pm}[z^{\pm}(\omega)]$, $\omega \in \Delta^{\pm}$. Then the Carleman problem (3.2) reduces to the Riemann problem

$$\Phi_1^{+}(\sigma) = A_0(\sigma)\Phi_1^{-}(\sigma) + A_1(\sigma), \quad \sigma \in \partial\Delta^{+}.$$

which has been studied in considerable detail.

The solvability of the conformal-gluing problem for a simply connected domain bounded by a Lyapunov curve, on the assumption that $\alpha \in \mathbf{C}^{2+\beta}(\Gamma)$, was proved by Mandzhavidze and Khvedelidze [23], and in that work they also established for the first time the conformal equivalence of the Carleman and Riemann problems. For multiply connected domains an analogous result was obtained in [24] for weaker assumptions concerning displacement; $\alpha \in \mathbf{C}^{1+\beta}(\Gamma)$. It should be noted that earlier, Volkovyskii [25] studied a special problem in conformal gluing for $\alpha \in \mathbf{C}^{1}(\Gamma)$.

Antontsev and Monakhov [10, 26] established the solvability of the above-mentioned quasiconformal-gluing problem for quasilinear elliptic equations (1.5) in the case of multiply connected domains, and on the basis of this result they studied the Carlemann problem and more general boundary-value problems when there was displacement in the boundary conditions. In this case the requirements on the smoothness of the boundary Γ and the displacement $\alpha(t)$ are much weaker: the connected components of Γ are assumed to be quasiconformal curves (images of a circle under quasiconformal mappings), and the $\alpha(t)$ are assumed to be limiting values of the quasiconformal mapping $\alpha:D^{+} \to D^{+}, |\alpha_{\bar{z}}/\alpha_z| \leqslant q_0 < 1$, which does not, in general, have derivatives on Γ.

How "bad" these quasiconformal curves are can be seen from the well-known example given by Ponomarev [27], who constructed in the neighborhoods $\Omega^{\pm} \subset D^{\pm}$ $(\overline{D^{+}} \cup D^{-} = \mathbf{C})$ of the curve Γ holomorphic functions $\omega^{\pm}(z)$ which can be continued on Γ to a function that is continuous but not holomorphic in $\Omega = (\Omega^{+} \cup \Omega^{-} \cup \Gamma)$; this function is $\omega(z) = \omega^{+}(z)$ $(z \in \Omega^{+})$ and $\omega = \omega^{-}(z)$ $(z \in \Omega^{-})$.

For nonlinear L-elliptic equations the problem of quasiconformal gluing has not been solved up to the present time.

4. Harmonic Mappings. In 1962-1967 Lavrent'ev laid the foundations of the theory of mappings of three-dimensional domains corresponding to system of partial differential equations. It should be noted that the connection between quasiconformal mappings of multidimensional euclidean spaces and the theory of differential equations is not so close as in the plane case.

On the basis of hydrodynamic considerations, Lavrent'ev distinguished two classes of mappings of three-dimensional domains in $\mathbf{R}^3$, which he called harmonic mappings.

The first class includes the mappings $x \to u(z)$ satisfying a system of four differential equations

$$\operatorname{rot} u = 0, \ \operatorname{div} u = 0. \tag{4.1}$$

In simply connected domains the solutions of (4.1) admit of the representation $u = \nabla\varphi$, in which φ is an arbitrary harmonic function. Harmonic mappings satisfy many theorems of the theory of functions of a complex variable. We shall formulate two of them. The first assertion is given in [28]:

THEOREM 3. Let $u \not\equiv 0$ be the solution of Eqs. (4.1) and suppose that the Jacobian $J(x)$ of the mapping $u(x)$ vanishes at the point x_0, which is in the interior of the domain of definition of u. Then in any neighborhood of x_0 the function $J(x)$ changes sign. An obvious consequence of this theorem is the following:

THEOREM 4. Suppose that the sequence u_n of solutions of (4.1) which are one-sheet functions in the ball $B_1:|x| < 1$ converges at every point of B_1 to the harmonic mapping u. Then u is also a one-sheet function in B_1.

The question of the existence of harmonic mappings of given domains was first considered by Lavrent'ev in 1967. In [29] he proved the solvability of the problem of the harmonic mapping of a layer $\{z_0(x_1, x_2) < x_3 < z_1(x_1, x_2)\}$ of the space $\mathbf{R}^3$ onto the layer $\{0 < u_3 < H\}$ lying in the space of points $u = (u_1, u_2, u_3)$, on the assumption that the functions z_1, which are three times continuously differentiable, will tend at an exponential rate to different constant values as $x_1^2 + x_2^2 \to \infty$. In [29, 30] he formulated a number of hypotheses concerning the existence of harmonic mappings of simply connected domains and, in particular, the possibility of harmonic mappings of a ball onto a three-axis ellipsoid and of a ball onto itself.

The solution of the last problem was given by Antontsev [31]. We give below a statement of the results. We shall denote by B_λ the family of ellipsoids of revolution

$$B_\lambda = \{x : x_1^2 + x_2^2 + \lambda x_3^2 < 1\}.$$

THEOREM 5. For every $\lambda \in [^1/_2, \infty)$ there exists a harmonic function $\varphi = \varphi(\sqrt{x_1^2 + x_2^2}, x_3)$ which belongs to the space $C^\infty(B_1)$ and whose gradient diffeomorphically maps the ball B_1 onto B_λ in a manner which preserves orientation.

The proof is based on the fact that when there is axial symmetry, the problem of the harmonic mapping of a ball onto a solid of revolution reduces to finding a quasiconformal mapping of a circular disk onto the intersection of this solid of revolution with a meridional plane.

It is not known at present whether there are any harmonic mappings of a ball even onto regions close to the ellipsoids B_λ. Therefore it is important to investigate the following problem concerning the stability of harmonic mappings.

We consider a one-parameter family of simply connected domains Ω_ε ($\varepsilon \geqslant 0$) with boundaries of the class C^∞ given by the equations $f_0(u) + \varepsilon f(u) = \text{const}$, and we shall suppose that the mapping $u_0 = \nabla\varphi_0$ ($\Delta\varphi_0 = 0$) of the ball B_1 onto Ω_0 is known.

The following question arises: what conditions must be satisfied by the family Ω_ε in order that for every ε belonging to an interval $[0, \varepsilon_0)$ there should exist a harmonic mapping u_ε of the ball B_1 onto the domain Ω_ε.

The solution of this problem reduces to finding a harmonic function $\varphi_\varepsilon(x)$ satisfying the boundary condition

$$f_0(\nabla\varphi_\varepsilon) + \varepsilon f(\nabla\varphi_\varepsilon) = \text{const} \quad \text{for} \quad |x| = 1.$$

In a first approximation with respect to ε, for the difference $\psi = \varphi_\varepsilon - \varphi_0$, we can obtain a boundary-value problem with the inclined derivative

$$a \cdot \nabla\psi = g \quad \text{for} \quad |x| = 1, \tag{4.2}$$

where

$$a = f'_{0,u}(\nabla\varphi_0); \quad g = -\varepsilon f(\nabla\varphi_0).$$

If problem (4.2) is nondegenerate (the product $a \cdot x$ does not vanish on the unit sphere), then the solution of the problem of the stability of the harmonic mapping is found by the method of successive approximation. Unfortunately, this case is rarely encountered; the following assertion (P. I. Plotnikov) holds: If the domain Ω_0 is convex, then there exists on the unit sphere a point x_0 such that $a(x_0) \cdot x_0 = 0$.

The proof is based on the fact that for the assumptions we have made, we have the representation

$$a \cdot x = b_{ij}\varphi_{0,x_i}x_j, \quad b_{ij}x_ix_j \leqslant 0,$$

which, together with the Zaremba–Hopf theorem, guarantees that the condition $a \cdot x = 0$ will be satisfied at least at one point x_0, $|x_0| = 1$.

This makes the problem very difficult. Nevertheless, it appears that a solution of the problem can be obtained by making use of the existing investigations of the problem with an inclined derivative [32].

5. Generalizations of the Cauchy–Riemann System. The second class of quasiconformal mappings of three-dimensional domains considered by Lavrent'ev is related to systems of third-order differential equations:

$$\nabla u_1 = \lambda(|\nabla u_1|)\nabla u_2 \times \nabla u_3. \tag{5.1}$$

On the basis of geometric considerations, Lavrent'ev proved in [29] that the system (5.1) is a natural generalization of the Cauchy–Riemann equations. The equations (5.1) admit of an obvious hydrodynamic interpretation. The function u_1 is the flow potential of a barotropic liquid and satisfies the equation

$$\mathrm{div}\,[\lambda^{-1}(|\nabla u_1|)\nabla u_1] = 0.$$

The components u_2, u_3 of u are constant on the trajectories of the dynamic system $\dot{x} = \nabla u_1$ and consequently serve as the streamlines of the flow generated by the potential u_1.

The first investigation of these mappings was given in [33], in which the author constructs a number of exact solutions of system (5.1) whose analogs are linear fractional, exponential, and logarithmic functions of a complex variable.

The problem of the mappings of domains of the three-dimensional layer type by the solutions of Eqs. (5.1) were considered by Plotnikov [34, 35]. We give one of the results he obtained.

Suppose that D is a layer in $\mathbf{R}^3$ which is bounded by smooth surfaces $S_0: x_3 = z_0(x_1, x_2)$ and $S_1: x_3 = z_0 + h(x_1, x_2)$. We assume that as $x_1 \to \pm\infty$, the surface S_0 tends at an exponential rate to the planes $x_3 = \tan\theta \cdot x_1$, $x_3 = 0$, and the "depth" h correspondingly tends to some limiting values $h^{\pm}(x_2)$. We fix a smooth mapping $v = (v_2, v_3)$ (preserving an element of area) of the strip $0 < x_3 < h^{-}(x_2)$ lying in the plane x_2, x_3 onto a rectilinear strip $0 < v_3 < H$. We consider in D the following boundary-value problem for the harmonic function u_1:

$$\begin{gathered}\nabla u_1 \cdot \mathbf{n} = 0 \text{ on } S_0 \cup S_1,\\ |x_1^3(\nabla u_1 - p^{\pm})| \to 0 \quad \text{as} \quad x_1 \to \pm\infty,\end{gathered} \tag{5.2}$$

where the unit vectors are $p^- = (1, 0, 0)$, $p^+ = (\cos\theta, 0, \sin\theta)$.

THEOREM 6. If problem (5.2) has a solution satisfying the inequality $|\ln|\nabla u_1|| \leqslant c_0 < \infty$, then there exists a unique solution of Eqs. (5.1) with $\lambda = 1$ (unique to within an additive constant) which homeomorphically maps D onto the layer $\{0 < u_3 < H\}$ and satisfies the normalization conditions

$$u_j \to v_j \text{ as } \quad x_1 \to -\infty, \; j = 2, 3.$$

The hypothesis of the theorem is satisfied, e.g., in the case when the derivatives z_{0,x_2}, h_{x_2} are sufficiently small.

6. Wave Theory. In 1957, in analyzing the phenomenon of tsunamis, Lavrent'ev advanced the hypothesis that an irregularity in the bottom of a body of water, such as an underwater ridge, could serve as a waveguide for surface waves. This problem gave rise to a number of investigations [36-39] carried out in the years 1959-1975 at the Hydrodynamics Institute of the Siberian Branch of the Academy of Sciences of the USSR. In 1965 R. M. Galineina confirmed the validity of Lavrent'ev's hypothesis within the framework of the linear theory of waves. We shall consider these results in more detail. Assume that an ideal incompressible liquid fills a domain $D \subset \mathbf{R}^3$ bounded by a "free surface" — the plane $\Gamma: x_3 = 0$ and the bottom $\Gamma_\varepsilon: x_3 = -1 + \varepsilon h(x_2)$. The finite infinitely differentiable function $h \geqslant 0$ defines the shape of an underwater ridge. The Cauchy—Poisson problem of nonstationary waves on the surface of an ideal liquid reduces to finding the flow potential — the harmonic function $\varphi(t, x)$ satisfying the boundary conditions

$$\varphi_{tt} + g\varphi_{x_3} = 0 \quad \text{on} \quad \Gamma, \nabla\varphi \cdot n = 0 \quad \text{on} \quad \Gamma_\varepsilon \tag{6.1}$$

and the initial conditions

$$\varphi|_{t=0} = \varphi_0, \; \varphi_t|_{t=0} = \varphi_1 \text{ on } \Gamma. \tag{6.2}$$

The wave process is accompanied by a waveguide effect if the problem (6.1) has a solution

$$\varphi = e^{i(\omega t - \nu x_1)}\Phi(x_2, x_3), \quad \Phi \to 0 \quad \text{as} \quad x_2 \to \infty. \tag{6.3}$$

In the general case it must be shown that the solution of the Cauchy—Poisson problem (6.1), (6.2) admits of the asymptotic representation

$$\varphi = \sum_{k=1}^{n} a_k(t)\,\varphi_k(t, x) + \varphi^*(t, x),$$

in which the φ_k have the form (6.3) and the remainder term φ^* is small in comparison with the a_k.

The proof of the existence of solutions of (6.1) in the form of waves traveling along an underwater ridge can be reduced [36, 39] to the proof of the existence of nontrivial solutions, damped at infinity, of the integrodifferential equation

$$\frac{d^2u}{dx_2^2} + \lambda u = \varepsilon A(\varepsilon, \lambda) u \qquad (6.4)$$

where $u(x_2)$ is the restriction of Φ to the plane Γ. Here A is self-adjoint compact operator in the scale of Sobolev spaces $s \geqslant 0$. In [39] it was established that the existence of negative eigenvalues of (6.4) and the corresponding rapidly decreasing eigenfunctions is a consequence of a general assertion for a broad class of compact operators which are not necessarily self-adjoint. In [37] it was shown that the terms corresponding to the eigenfunctions of (6.4) in the asymptotic solution of the Cauchy–Poisson problem (1.7), (2.1) are $a_k\varphi_k$ with $a_k \sim t^{\alpha_k}$, $\alpha_k > -1$. The question of estimating the remainder term φ^* which results from the continuous spectrum of (6.4) remains unanswered. We should expect, by analogy with the case of a smooth bottom ($\varepsilon = 0$), considered in [40], that along the ridge $\varphi^* \sim t^{-1}$, but no rigorous proof of this is available.

7. Three-Dimensional Flows with Free Boundaries. The theory of three-dimensional flows with free boundaries is a branch of hydrodynamics which has not been extensively developed thus far. Its basic outlines were given by Lavrent'ev in 1962-1968. In [41] he considered a three-dimensional variant of Kirchhoff's problem concerning the potential motion of a layer of incompressible liquid with a free boundary and proposed some methods for solving it. In [2] Lavrent'ev and Shabat formulated a number of problems concerning three-dimensional waves and jet flow.

The correctness of the formulation of the problem of three-dimensional potential flows of an ideal liquid was first investigated in [42]. It was found that this problem is degenerate and its formulation must be determined by the geometry of the flow. In [43] Plotnikov solved a problem stated by Lavrent'ev [2] concerning three-dimensional gravitational waves on the surface of a layer of ideal incompressible liquid. The problem reduces to finding a free surface S with the graph

$$x_3 = \varepsilon \cos k_1x_1 \cos k_2x_2 + \eta(\varepsilon, x_1, x_2) \equiv Z$$

and a flow potential which is a function φ harmonic in the layer $\{-h < x_3 < Z\}$ and satisfies the equations

$$\varphi_{x_3} = 0 \text{ for } x_3 = -h;\ |\nabla\varphi|^2 + \lambda_1 x_3 = \lambda_2 \text{ on } S, \qquad \nabla\varphi\cdot n = 0 \text{ on } S, \qquad (7.1)$$

where the λ_i are parameters. The solution of problem (7.1) should be sought in the class of smooth 2π-periodic functions η, $x_1 - \varphi$ in the variables x_1, x_2. The fundamental difference between this problem and the analogous two-dimensional problems of wave theory consists in the fact that the eigenvalues of the three-dimensional linear problem of wave theory — the linearization of (7.1) on the trivial solution $Z = \text{const}$, $\varphi = x_1$ — are everywhere dense on the positive semiaxis. Using the method of accelerated convergence, the following assertion was proved in [43].

Theorem 7. In the space of parameters λ_i, h there exists a nonempty set Λ at every point of which there branches off from the trivial solution Z = const a one-parameter family of solutions of the problem $\varphi(\varepsilon)$, $\eta(\varepsilon)$, $\lambda_i(\varepsilon)$. The following estimate holds:

$$\|\eta(\varepsilon)\|_{C^r(\mathbb{R}^2)} \leqslant \text{const}\, \varepsilon^2, \quad r \geqslant 3.$$

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SUPPRESSION OF TURBULENCE IN THE CORES OF CONCENTRATED VORTICES

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1. The problem of the motion of vortex rings has intrigued researchers for more than a century now [1]. On the initiative of M. A. Lavrent'ev, the Institute of Hydrodynamics of the Siberian Branch of the Academy of Sciences of the USSR has been conducting experimental and theoretical studies for several years on this effect and other rotational flows of liquids and gases [2]. A mathematical model for the description of the motion of turbulent vortex rings has been proposed on the basis of an analysis of the experimental facts [3, 4]. This model rests on the hypothesis that the turbulent nature of the motion and the transport of a tracer impurity by it can be described by means of scalar coefficients of turbulent viscosity ν and turbulent diffusion $\varkappa$ that vary with time but do not depend on the space coordinates. The additional assumption of flow self-similarity, which is highly consistent with the experimental results, has made it possible to calculate the structure of a vortex ring in the vanishing-viscosity limit [5]; the theory in this case does not contain any empirical constants. However, a comparison of the calculations with the existing experimental results discloses a significant discrepancy in the vicinity of the core of the vortex ring.

It is now clear that the principal cause of this discrepancy is the assumption of turbulence "uniformity" throughout the vortex volume. The results of qualitative experiments and certain theoretical considerations [6] indicate that the core of the vortex ring is almost completely devoid of turbulent tracer transport (the "laminar core" effect) in connection with strong turbulent tracer transport in the atmosphere of the vortex ring. This turbulence suppression is caused by the presence in rapidly rotating flow of a singular "elasticity" associated with the gyroscopic behavior of the fluid particles.

The unsteadiness of the flow in the vortex makes it exceedingly difficult to study the effect discovered in [6]. In this article we describe experiments by which it is possible to observe a similar effect under steady-state conditions; we also present a qualitative explanation of the effect and propose simple models of the turbulent stresses and tracer transport in the cores of line and ring vortices.

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